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MULTIPLIERS ON SPACES OF FUNCTIONS ON A LOCALLY COMPACT ABELIAN GROUP WITH VALUES IN A HILBERT SPACE

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We prove a representation theorem for bounded operators commuting with translations on $L^2_\omega(G, H)$, where G is a locally compact abelian group, H is a Hilbert space and ω is a weight on G . Moreover, in the particular case when $G = \mathbb{R}$, we characterize completely the spectrum of the shift operator $S_{1,\omega}$ on $L^2_\omega(\mathbb{R}, H)$.

1. Introduction. Let G be a locally compact abelian group. Denote by \widehat{G} the dual group of G . The groups G and \widehat{G} are equipped with the Haar measure. Let H be a separable Hilbert space and denote by $\langle u, v \rangle$ the scalar product of two elements u and v in H . Let ω be a weight on G i.e. ω is a continuous, positive, measurable function on G such that

$$0 < \sup_{x \in G} \frac{\omega(x+y)}{\omega(x)} < +\infty, \quad \forall y \in G.$$

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For $1 \leq p < +\infty$, we denote by $L_\omega^p(G, H)$ the space of the functions f on G with values in H such that

$$G \ni x \longrightarrow \|f(x)\| \in \mathbb{R}^+$$

is a function in $L_\omega^p(G)$, where $L_\omega^p(G)$ is the set of the measurable functions g on G such that

$$\int_G |g(x)|^p \omega(x) dx < +\infty.$$

Let $C_c(G)$ be the space of the continuous functions from G into \mathbb{C} with compact support. Denote by $C_c(G, H)$ the space of the functions f on G with values in H such that $\|f(\cdot)\| \in C_c(G)$. For $a \in G$, define

$$S_{a,\omega} : L_\omega^p(G, H) \longrightarrow L_\omega^p(G, H)$$

by the formula

$$(S_{a,\omega}f)(x) = f(x - a), \quad \forall f \in L_\omega^p(G, H), \quad a.e.$$

and let

$$\mathbf{S}_{a,\omega} : L_\omega^p(G) \longrightarrow L_\omega^p(G)$$

be the operator defined by the formula

$$(\mathbf{S}_{a,\omega}g)(x) = g(x - a), \quad \forall g \in L_\omega^p(G), \quad a.e.$$

Notice that we have

$$\|S_{a,\omega}\| = \|\mathbf{S}_{a,\omega}\| = \sup_{x \in G} \frac{\omega(x + a)}{\omega(x)}, \quad \forall a \in G,$$

and consequently

$$\rho(S_{a,\omega}) = \rho(\mathbf{S}_{a,\omega}), \quad \forall a \in G.$$

Here $\rho(S_{a,\omega})$ (resp. $\rho(\mathbf{S}_{a,\omega})$) denotes the spectral radius of $S_{a,\omega}$ (resp. $\mathbf{S}_{a,\omega}$). Denote by $C_c(G) \otimes H$ the closed vector space generated by functions

$$fu : G \ni x \longrightarrow f(x)u \in H$$

with $f \in C_c(G)$ and $u \in H$. The space $C_c(G) \otimes H$ is dense in $L_\omega^p(G, H)$, for $1 \leq p < +\infty$. We say that M is a multiplier on $L_\omega^p(G, H)$ if M is a bounded operator from $L_\omega^p(G, H)$ into $L_\omega^p(G, H)$ such that

$$MS_a = S_aM, \quad \forall a \in G.$$

Define \mathcal{M}_ω^p the algebra of the multipliers on $L_\omega^p(G, H)$. We denote by \mathcal{F} (resp. \mathcal{F}) the usual Fourier transformation from $L^2(G, H)$ (resp. $L^2(G)$) into $L^2(\widehat{G}, H)$ (resp. $L^2(\widehat{G})$). We have the following representation theorem for the multipliers on $L^p(G, H)$.

Theorem 1 ([2]). *For every M multiplier on $L^p(G, H)$, $1 \leq p < +\infty$, there exists a measurable function*

$$\Phi_M : \widehat{G} \longrightarrow \mathcal{L}(H),$$

which is essentially bounded for the operator norm of $\mathcal{L}(H)$ such that

$$\mathcal{F}(Mf)(\chi) = \Phi_M(\chi)[\mathcal{F}(f)(\chi)], \text{ a.e. on } \widehat{G},$$

for every $f \in L^p(G, H) \cap L^2(G, H)$. Moreover,

$$\operatorname{ess\,sup}_{\chi \in \widehat{G}} \|\Phi_M(\chi)\| \leq \|M\|.$$

The proof of this theorem is based on the well-known result about the multipliers on $L^p(G)$. Indeed, for every bounded operator M commuting with the translations on $L^p(G)$ there exists a function $h \in L^\infty(\widehat{G})$ (see [3]) such that

$$(1.1) \quad \widehat{Mf} = h\hat{f}, \quad \forall f \in C_c(G)$$

and $\|h\|_\infty \leq \|M\|$. This paper is motivated by a recent result generalizing the representation (1.1) for a more general class of Banach spaces of functions on G . The spaces $L_\omega^p(G)$ are included in this class. Denote by \widetilde{G}_ω^p the set of the continuous morphisms θ from G into \mathbb{C}^* such that

$$(1.2) \quad \left| \int_G f(x) \theta^{-1}(x) dx \right| \leq \|M_f\|_{\mathcal{L}(L_\omega^p(G))},$$

where M_f is the operator of convolution by f on $L_\omega^p(G)$. Define

$$\widetilde{G}_\omega^{p+} = \{|\theta|, \theta \in \widetilde{G}_\omega^p\}.$$

It was proved in [6] that the set $\widetilde{G}_\omega^{p+}$ is not empty, log-convex and compact for the topology of the uniform convergence on every compact set of G . It is clear that $\widetilde{G}_\omega^p = \widetilde{G}_\omega^{p+} \widehat{G}$. Let \widetilde{G} be the set of the continuous morphisms from G into \mathbb{C}^* . We have the following proposition.

Proposition 1 (see [6], [7]). *If G is either a discrete group or a compact group, we have*

$$\widetilde{G}_\omega^p = \{\theta \in \widetilde{G} \mid |\theta^{-1}(x)| \leq \rho(S_{x,\omega}), \forall x \in G\}$$

and \widetilde{G}_ω^p is isomorphic to the joint spectrum of $\{S_{x,\omega}\}_{x \in G}$. The same result holds for $G = \mathbb{R}$.

Also in [6], was proved the following result, which we will use.

Theorem 2 ([6], [7]). *Fix $\theta \in \widetilde{G}_\omega^p$. For every bounded operator M commuting with the translations on $L_\omega^p(G)$, we have $(Mg)\theta^{-1} \in L^2(G)$, $\forall g \in C_c(G)$. There exists a function $h_{M,\theta} \in L^\infty(\widehat{G})$ such that*

$$\widehat{(Mg)\theta^{-1}} = h_{M,\theta} \widehat{(g\theta^{-1})}, \forall g \in C_c(G)$$

and $\|h_{M,\theta}\|_\infty \leq C_\omega \|M\|$, where C_ω is a constant independent of M .

The main result in this paper is the following.

Theorem 3. *For $M \in \mathcal{M}_\omega^p$ and $\theta \in \widetilde{G}_\omega^p$, we have:*

- 1) $(Mg)\theta^{-1} \in L^2(G, H)$, $\forall g \in C_c(G) \otimes H$.
- 2) *There exists $\Phi_\theta \in L^\infty(\widehat{G}, \mathcal{L}(H))$ such that*

$$\mathcal{F}((Mg)\theta^{-1})(\chi) = \Phi_\theta(\chi)[\mathcal{F}(g\theta^{-1})(\chi)], \forall g \in C_c(G) \otimes H, \text{ a.e.}$$

Moreover, $\text{ess sup}_{\chi \in \widehat{G}} \|\Phi_\theta(\chi)\| \leq C_\omega \|M\|$.

2. Proof of Theorem 3. Since Theorem 2 plays an important role in the proof of Theorem 3, for the convenience of the reader we give a sketch of it's proof. The full proof is exposed in [7] and [6].

Proof of Theorem 2. First, every multiplier M in $L_\omega^p(G)$ is the limit for the strong operators topology of a net (M_{ϕ_α}) where $\phi_\alpha \in C_c(G)$ and $\|M_{\phi_\alpha}\| \leq C_\omega \|M\|$. This may be proved using the fact that the restriction of every multiplier on $C_c(G)$ is a convolution with a quasimeasure (see [2]). Fix $M \in \mathcal{M}_\omega^p$ and let (M_{ϕ_α}) be a net satisfying the above property. Fix $\theta \in \widetilde{G}_\omega^p$. From the definition of \widehat{G} , it follows that

$$\left| \widehat{\phi_\alpha \theta^{-1}}(\chi) \right| \leq \|M_{\phi_\alpha}\| \leq C_\omega \|M\|, \forall \chi \in \widetilde{G}_\omega^p.$$

If we replace (ϕ_α) by a suitable subnet, we obtain that $(\widehat{\phi_\alpha \theta^{-1}})$ converges to a function $h_{M,\theta} \in L^\infty(\widehat{G})$ for the weak* topology $\sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$. This implies that for each $f \in C_c(G)$, the net

$$\left(\mathcal{F}((M_{\phi_\alpha} f) \theta^{-1}) \right) = \left(\mathcal{F}((\phi_\alpha * f) \theta^{-1}) \right) = \left(\widehat{\phi_\alpha \theta^{-1} f \theta^{-1}} \right)$$

converges to $h_{M,\theta} \widehat{f \theta^{-1}}$ with respect to the weak topology of $L^2(\widehat{G})$. Consequently,

$$\lim_{\alpha} (M_{\phi_\alpha} f) \theta^{-1} = \mathcal{F}^{-1}(h_{M,\theta} \widehat{f \theta^{-1}}), \quad \forall f \in C_c(G)$$

with respect to the weak topology of $L^2(G)$. On the other hand, since $L_\omega^p(G) \subset L_{loc}^1(G)$ and the inclusion is continuous, for $g \in C_c(G)$, we get

$$\lim_{\alpha} \left| \int_G g(y) \theta^{-1}(y) (M_{\phi_\alpha} f(y) - M f(y)) dy \right| = 0.$$

We conclude that for every $f \in C_c(G)$ the functions $(M f) \theta^{-1}$ and $\mathcal{F}^{-1}(h_{M,\theta} \widehat{f \theta^{-1}})$ define the same linear functional on $C_c(G)$ and so $(M f) \theta^{-1}(x) = \mathcal{F}^{-1}(h_{M,\theta} \widehat{f \theta^{-1}})(x)$, for almost every $x \in G$. We conclude that $(M f) \theta^{-1} \in L^2(G)$ and

$$(\widehat{M g}) \theta^{-1} = h_{M,\theta} \widehat{(g \theta^{-1})}, \quad \forall g \in C_c(G). \quad \square$$

In order to proof Theorem 3, we need the following lemma.

Lemma 1. *Let $g \in L^2(G, H)$ and $v \in H$. Then we have*

$$\mathcal{F}(\langle g(\cdot), v \rangle)(\chi) = \langle \mathcal{F}(g)(\chi), v \rangle,$$

for almost every $\chi \in \widehat{G}$.

Proof. Let $g \in L^2(G, H)$ and $(g_n)_{n \in \mathbb{N}} \subset C_c(G, H)$ be a sequence converging to g in $L^2(G, H)$. Then, we have

$$\mathcal{F}(\langle g(\cdot), v \rangle) = \lim_{n \rightarrow +\infty} \mathcal{F}(\langle g_n(\cdot), v \rangle),$$

with respect to the norm of $L^2(\widehat{G})$. For fixed $\chi \in \widehat{G}$ and $v \in H$, the map

$$C_c(\widehat{G}, H) \ni h \longrightarrow \langle h(\chi), v \rangle \in \mathbb{C}$$

is a continuous linear form. For given $\phi \in C_c(\widehat{G})$, the integral

$$\int_G g_n(x) \phi(\chi) \chi^{-1}(x) dx$$

is a convergent Bochner integral with values in $C_c(\widehat{G}, H)$. Indeed, we have

$$\begin{aligned} & \int_G \sup_{\chi \in \widehat{G}} \|g_n(x) \phi(\chi) \chi^{-1}(x)\| dx \\ & \leq \|\phi\|_\infty \int_G \|g_n(x)\| dx < +\infty. \end{aligned}$$

Since the Bochner integral commutes with continuous linear maps, we have for almost every $\chi \in \widehat{G}$,

$$\phi(\chi) \mathcal{F}(\langle g_n(\cdot), v \rangle)(\chi) = \phi(\chi) \langle \mathcal{F}(g_n)(\chi), v \rangle.$$

Since $\lim_{n \rightarrow +\infty} \mathcal{F}(g_n) = \mathcal{F}(g)$ with respect to the norm of $L^2(\widehat{G}, H)$, if we replace $(g_n)_{n \in \mathbb{N}}$ by a suitable subsequence, we get

$$\lim_{n \rightarrow +\infty} \|\mathcal{F}(g_n)(\chi) - \mathcal{F}(g)(\chi)\| = 0, \text{ a.e.}$$

and hence

$$\lim_{n \rightarrow +\infty} \langle \mathcal{F}(g_n)(\chi), v \rangle = \langle \mathcal{F}(g)(\chi), v \rangle, \text{ a.e.}$$

We conclude that

$$\mathcal{F}(\langle g(\cdot), v \rangle)(\chi) = \langle \mathcal{F}(g)(\chi), v \rangle,$$

for almost every $\chi \in \widehat{G}$. \square

Proof of Theorem 3. Fix $M \in \mathcal{M}_\omega^p$ and fix u and $v \in H$. Introduce the operator $M_{u,v}$ defined for $f \in L_\omega^p(G)$ by the formula

$$(2.1) \quad M_{u,v}(f)(x) = \langle M(fu)(x), v \rangle, \text{ a.e.}$$

Notice that $M_{u,v}(f) \in L_\omega^p(G)$ for every $f \in L_\omega^p(G)$. Indeed, since $M(fu) \in L_\omega^p(G, H)$, we have

$$\begin{aligned} & \int_G |\langle M(fu)(x), v \rangle|^p \omega(x)^p dx \\ & \leq \int_G \|M(fu)(x)\|^p \|v\|^p \omega(x)^p dx < +\infty. \end{aligned}$$

Moreover, notice that

$$\|M_{u,v}\| \leq \|M\| \|u\| \|v\|.$$

It is clear that

$$\langle M(S_a(fu))(x), v \rangle = \langle M(fu)(x - a), v \rangle, \text{ a.e.}$$

hence $M_{u,v}$ is a multiplier on $L_\omega^p(G)$. From Theorem 2, we obtain for every $\theta \in \widetilde{G}_\omega^p$,

$$(2.2) \quad (M_{u,v}f)\theta^{-1} \in L^2(G), \quad \forall f \in C_c(G)$$

and there exists $\Phi_{\theta,u,v} \in L^\infty(\widehat{G})$ such that

$$(2.3) \quad \mathcal{F}((M_{u,v}f)\theta^{-1})(\chi) = \Phi_{\theta,u,v}(\chi)\mathcal{F}(f\theta^{-1})(\chi), \quad a.e.$$

Let \mathcal{O} be a countable orthonormal basis of H and let F be the set of finite linear combinations of elements of \mathcal{O} . We have

$$|\Phi_{\theta,u,v}(\chi)| \leq C_\omega \|M_{u,v}\|, \quad \forall \chi \in \widehat{G} \setminus N_{u,v},$$

where $N_{u,v}$ is a set of measure zero. Without loss of generality, we can modify $\Phi_{\theta,u,v}$ on $N = \cup_{(u,v) \in F \times F} N_{u,v}$ in order to obtain

$$|\Phi_{\theta,u,v}(\chi)| \leq C_\omega \|M_{u,v}\| \leq C_\omega \|M\| \|u\| \|v\|, \quad \forall u, v \in F, a.e.$$

For fixed $\chi \in \widehat{G} \setminus N$

$$F \times F \ni (u, v) \longrightarrow \Phi_{\theta,u,v}(\chi) \in \mathbb{C}$$

is a sesquilinear and continuous form on $F \times F$ and since F is dense in H , we conclude that there exists a unique map

$$H \times H \ni (u, v) \longrightarrow \widetilde{\Phi}_{\theta,u,v}(\chi) \in \mathbb{C}$$

such that

$$\widetilde{\Phi}_{\theta,u,v}(\chi) = \Phi_{\theta,u,v}(\chi), \quad \forall u, v \in F.$$

Consequently, there exists a unique map

$$\Phi_\theta : \widehat{G} \longrightarrow \mathcal{L}(H)$$

such that

$$\langle \Phi_\theta(\chi)[u], v \rangle = \widetilde{\Phi}_{\theta,u,v}(\chi), \quad \forall u, v \in H.$$

It is clear that

$$\|\Phi_\theta(\chi)\| = \sup_{\|u\|=1, \|v\|=1} |\langle \Phi_\theta(\chi)[u], v \rangle| \leq C_\omega \|M\|, \quad a.e.$$

Fix $\theta \in \widetilde{G}_\omega^p$ and $f \in C_c(G)$. For every $\chi \in \widehat{G}$, we have $\widehat{f\theta^{-1}}(\chi)u \in H$. Next for almost every $\chi \in \widehat{G}$, we obtain

$$\langle \Phi_\theta(\chi)[\widehat{f\theta^{-1}}(\chi)u], v \rangle = \langle \Phi_\theta(\chi)[u], v \rangle \widehat{f\theta^{-1}}(\chi)$$

$$\begin{aligned}
&= \Phi_{\theta,u,v}(\chi) \widehat{f\theta^{-1}}(\chi) = \mathcal{F}((M_{u,v}f)\theta^{-1})(\chi) \\
&= \mathcal{F}(\langle M[fu], v \rangle \theta^{-1})(\chi).
\end{aligned}$$

Consequently,

$$(2.4) \quad \mathcal{F}^{-1}(\langle \Phi_{\theta}(\cdot) [\widehat{f\theta^{-1}}(\cdot)u], v \rangle)(x) = \langle M[fu](x), v \rangle \theta^{-1}(x),$$

for almost every $x \in G$. Now, consider the function Ψ_{θ} on \widehat{G} defined for almost every $\chi \in \widehat{G}$ by the formula

$$\Psi_{\theta}(\chi) = \Phi_{\theta}(\chi) [\widehat{f\theta^{-1}}(\chi)u]$$

and observe that $\Psi_{\theta} \in L^2(\widehat{G}, H)$. Indeed, we have

$$\begin{aligned}
&\int_{\widehat{G}} \|\Phi_{\theta}(\chi) [\widehat{f\theta^{-1}}(\chi)u]\|^2 d\chi \\
&\leq \int_{\widehat{G}} \|\Phi_{\theta}(\chi)\|^2 \|\widehat{f\theta^{-1}}(\chi)u\|^2 d\chi \\
&\leq C_{\omega}^2 \|M\|^2 \int_{\widehat{G}} |\widehat{f\theta^{-1}}(\chi)|^2 \|u\|^2 d\chi < +\infty.
\end{aligned}$$

This makes possible to apply Lemma 1, and we get

$$\mathcal{F}^{-1}(\langle \Phi_{\theta}(\cdot) [\widehat{f\theta^{-1}}(\cdot)u], v \rangle)(x) = \langle \mathcal{F}^{-1}(\Phi_{\theta}(\cdot) [\widehat{f\theta^{-1}}(\cdot)u])(x), v \rangle,$$

for almost every $x \in G$. It follows from (2.4) that we have

$$M[fu](x) \theta^{-1}(x) = \mathcal{F}^{-1}(\Phi_{\theta}(\cdot) [\widehat{f\theta^{-1}}(\cdot)u])(x),$$

for almost every $x \in G$ and this yields

$$M[fu] \theta^{-1} \in L^2(G, H).$$

Moreover, we obtain

$$\mathcal{F}(M[fu] \theta^{-1})(\chi) = \Phi_{\theta}(\chi) [\widehat{f\theta^{-1}}(\chi)u],$$

for almost every $\chi \in \widehat{G}$. \square

3. The case $G = \mathbb{R}$. In [4] we have established a more complete version of Theorem 2 concerning multipliers on weighted spaces on \mathbb{R} . Let w be a weight on \mathbb{R} and denote by S_ω the operator $S_{1,\omega}$ on $L_\omega^2(\mathbb{R}, H)$. Define

$$I_\omega = [-\ln \rho(S_\omega^{-1}), \ln \rho(S_\omega)]$$

and

$$\Omega_\omega = \{z \in \mathbb{C}, \operatorname{Im} z \in I_\omega\}.$$

For $f \in L_\omega^2(\mathbb{R}, H)$ denote by $(f)_a$ the function

$$(f)_a(x) = f(x)e^{ax}, \forall a \in I_\omega.$$

We have the following theorem.

Theorem 4 ([4]). *Let ω be a weight on \mathbb{R} and let M be a multiplier on $L_\omega^2(\mathbb{R})$.*

i) We have $(Mf)_a \in L^2(\mathbb{R})$, $\forall f \in C_c^\infty(\mathbb{R})$ and there exists $h_a \in L^\infty(\mathbb{R})$ such that

$$\widehat{(Mf)_a}(x) = h_a(x)\widehat{(f)_a}(x), \forall a \in I_\omega, \forall f \in C_c^\infty(\mathbb{R}), \text{ a.e.}$$

and

$$\|h_a\|_\infty \leq C_\omega \|M\|.$$

ii) If $\mathring{\Omega}_\omega \neq \emptyset$, then there exists $h \in \mathcal{H}^\infty(\mathring{\Omega}_\omega)$, such that for every $f \in C_c^\infty(\mathbb{R})$,

$$\widehat{Mf}(z) = h(z)\hat{f}(z), \forall z \in \mathring{\Omega}_\omega,$$

where

$$\widehat{Mf}(x + ia) = \widehat{(Mf)_a}(x), \forall x + ia \in \mathring{\Omega}_\omega.$$

Using the same methods as those exposed in Section 2 combined with Theorem 4, we obtain the following interesting version of Theorem 3 in the particular case $G = \mathbb{R}$.

Theorem 5. *Let ω be a weight on \mathbb{R} . Let M be a multiplier on $L_\omega^2(\mathbb{R}, H)$. Then*

i) We have $(Mf)_a \in L^2(\mathbb{R}, H)$, $\forall f \in C_c(\mathbb{R}) \otimes H$ and there exists $\Phi_a \in L^\infty(\mathbb{R}, \mathcal{L}(H))$ such that

$$\widehat{(Mf)_a}(x) = \Phi_a(x)[\widehat{(f)_a}(x)], \forall a \in I_\omega, \forall f \in C_c(\mathbb{R}) \otimes H, \text{ a.e.}$$

and

$$\operatorname{ess\,sup}_{x \in \mathbb{R}} \|\Phi_a(x)\| \leq C_\omega \|M\|.$$

ii) If $\mathring{\Omega}_\omega \neq \emptyset$, then there exists

$$\Phi : \mathring{\Omega}_\omega \longrightarrow \mathcal{L}(H)$$

such that for every $f \in C_c(\mathbb{R}) \otimes H$,

$$\widehat{Mf}(z) = \Phi(z)[\hat{f}(z)], \quad \forall z \in \mathring{\Omega}_\omega,$$

where

$$\widehat{Mf}(x + ia) = (\widehat{Mf})_a(x), \quad \forall x + ia \in \mathring{\Omega}_\omega.$$

For every $u, v \in H$ the function

$$z \longrightarrow \langle \Phi(z)[u], v \rangle$$

is in $\mathcal{H}^\infty(\mathring{\Omega}_\omega)$.

Since the proof of Theorem 5 is very similar to that of Theorem 3, we omit the details. Notice that following the results of [4] and [6], if $G = \mathbb{R}$, the set \widehat{G}_ω^p given by (1.2), that we use in Theorem 3 is isomorphic to the strip Ω_ω and the set \widehat{G}_ω^{p+} is isomorphic to the segment I_ω . Applying Theorem 5, we get the following proposition.

Proposition 2. *Let ω be a weight on \mathbb{R} . We have*

$$\operatorname{spec}(S_\omega) = \left\{ z \in \mathbb{C}, \frac{1}{\rho(S_\omega^{-1})} \leq |z| \leq \rho(S_\omega) \right\}.$$

Proof. Let $\alpha \notin \operatorname{spec}(S_\omega)$. Then $M = (S_\omega - \alpha I)^{-1}$ is a multiplier. Applying Theorem 5, we get that for every $a \in I_\omega$, there exists $\Phi_a \in L^\infty(\mathbb{R}, H)$ such that

$$(\widehat{Mf})_a(x) = \Phi_a(x)[(\widehat{f})_a(x)], \quad \forall a \in I_\omega, \quad \forall f \in C_c(\mathbb{R}) \otimes H, \quad a.e.$$

and

$$\operatorname{ess\,sup}_{x \in \mathbb{R}} \|\Phi_a(x)\| \leq C_\omega \|M\|.$$

Replacing f by $(S_\omega - \alpha I)^{-1}g$ in the above formula, we obtain

$$\widehat{(g)_a}(x) = \Phi_a(x) \left[\mathcal{F} \left(((S_\omega - \alpha I)g)_a \right) (x) \right], \forall g \in C_c(\mathbb{R}) \otimes H, \forall a \in I_\omega, a.e.$$

We have

$$\begin{aligned} \mathcal{F} \left(((S_\omega - \alpha I)g)_a \right) (x) &= \int_G (g(t-1) - \alpha g(t)) e^{at} e^{-itx} dt \\ &= \widehat{(g)_a}(x) (e^{-ix} e^a - \alpha), \forall g \in C_c(\mathbb{R}) \otimes H, \forall a \in I_\omega, a.e. \end{aligned}$$

Consequently, we get

$$\Phi_a(x) [\widehat{(g)_a}(x)] = \frac{1}{e^{-ix} e^a - \alpha} \widehat{(g)_a}(x), a.e.$$

and hence

$$\|\Phi_a(x)\| \geq \frac{1}{e^{-ix} e^a - \alpha}, a.e$$

This shows that $e^a \neq |\alpha|$, for every $a \in I_\omega$ and from the definition of I_ω it follows that

$$\alpha \notin \left\{ z \in \mathbb{C}, \frac{1}{\rho(S_\omega^{-1})} \leq |z| \leq \rho(S_\omega) \right\}.$$

We deduce that

$$\left\{ z \in \mathbb{C}, \frac{1}{\rho(S_\omega^{-1})} \leq |z| \leq \rho(S_\omega) \right\} \subset \text{spec}(S_\omega)$$

and this completes the proof. \square

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